On constants related to the choice of the local time at 0, and the corresponding Itô measure for Bessel processes with dimension d=2(1-alpha), 0 < alpha < 1

Catherine Donati-Martin, Bernard Roynette, Pierre P. Vallois, Marc Yor

On constants related to the choice of the local time at 0, and the corresponding Itô measure for Bessel processes with dimension $d = 2(1-\alpha)$, $0 < \alpha < 1$.

Catherine Donati-Martin(1), Bernard Roynette(2), Pierre Vallois(2), Marc Yor(1),(3)

April 11, 2006

(1) Laboratoire de Probabilités et Modèles Aléatoires,
Universités Paris VI et VII, 175 rue du Chevaleret,
F-75013 Paris
donati@ccr.jussieu.fr

(2) Université Henri Poincaré, Institut Elie Cartan, BP239,
F-54506 Vandoeuvre-les-Nancy Cedex
roynette@iecn.u-nancy.fr
vallois@iecn.u-nancy.fr

(3) Institut Universitaire de France

Abstract The precise choice of the local time at 0 for a Bessel process with dimension $d \in]0, 2[$ plays some role in explicit computations or limiting results involving excursion theory for these processes. Starting from one specific choice, and deriving the main related formulae, it is shown how the various multiplicative constants correponding to other choices made in the literature enter into these formulae.

Key words : Bessel processes, excursion theory, local time.

AMS Classification : 60G44, 60G52, 60J25, 60J35, 60J55, 60J60, 60J65.

1 Introduction

1.1 Nowadays, it is no longer necessary to emphasize how powerful excursion theory for, say, the study of linear Brownian motion, is. This is due, in particular, to the several fundamental descriptions of Itô's characteristic measure of (Brownian) excursions, $\underline{n}_{1/2}$, which are mainly due to D. Williams ([17],§II.67).

The following compensation formula holds :

$$E\left[\sum_{\gamma\in G} H(\gamma, e_\gamma)\,1_{(\delta-\gamma>0)}\right] = E\left[\int_0^\infty dL_s \int_{\Omega_{exc}} \underline{n}_{1/2}\,(d\varepsilon) H(s,\varepsilon)\right] \tag{1.1}$$

where :

- $H : \mathbb{R}_+ \times \Omega \times \Omega_{exc} \to \mathbb{R}_+$ is $\mathcal{P}\otimes\mathcal{E}$ measurable, $\mathcal{P}$ denoting the predictable σ-field associated to the Brownian motion $(B_u, u \geq 0)$, and $(\Omega_{exc}, \mathcal{E})$ denoting the measurable space of generic excursions ϵ.
- $\{e_\gamma, \gamma \in G\}$ denotes the family of Brownian excursions, labelled with their starting time γ, and where δ denotes the ending time of the same excursion, i.e : $e_\gamma(u) = B_{\gamma+u}1_{(u\leq\delta-\gamma)}$
- G is the set of left extremities of maximal intervals $]\gamma,\delta[$ which constitute the complement of the random set $Z_w = \{s : B_s(w) = 0\}$; $e_\gamma(u) = B_{\gamma+u}1_{(u\leq\delta-\gamma)}$ denotes the excursion on $]\gamma,\delta[$.
- $(L_s, s\geq 0)$ is the standard local time at 0 of $(B_s, s \geq 0)$, i.e. : it satisfies

$$\{|B_s| - L_s, s\geq 0\} \quad \text{is again a Brownian motion.} \tag{1.2}$$

Formula (1.1) is a key formula from which many consequences may be derived. See, e.g, [[15] : Chap XII].
Note that on the RHS of (1.1), it is the product :

$$dL_s\,\underline{n}_{1/2}\,(d\varepsilon) \tag{1.3}$$

which appears, but since most authors take (1.2) as a definition of (L_s), then, as (1.3) is intrinsic, there is no ambiguity about the choice of $\underline{n}_{1/2}$.

1.2 This clear cut Brownian situation is no longer so unambiguous when one considers the excursion theory for other diffusions, and also Lévy processes. The aim of the present note is to give an easy access to pairs $(L, \underline{n}_\alpha)$ related to the d-dimensional Bessel process $(R_t, t\geq 0)$, starting from 0, with dimension $d = 2(1-\alpha)$ with $0 < d < 2$ or equivalently, with index $-\alpha = \frac{d}{2} - 1$ $(0 < \alpha < 1)$. To be precise, this process, which we shall denote sometimes as BES$(-\alpha)$, is an $\mathbb{R}_+$-valued Feller diffusion whose infinitesimal generator $\mathcal{L}$ is defined by :

$$\mathcal{L}f(r) = \frac{1}{2}\frac{d^2 f}{dr^2}(r) + \frac{1-2\alpha}{2r}\frac{df}{dr}(r)$$

on the domain :

$$\mathcal{D} = \left\{f : \mathbb{R}_+ \to \mathbb{R}\,;\, \mathcal{L}f \in C_b(\mathbb{R}_+)\,,\, \lim_{r\to 0} r^{1-2\alpha} f'(r) = 0\right\}.$$

From now on, this dimension d will be fixed, hence so will α.
Inspired by the Brownian case, and formula (1.2), we shall take as a definition of $(L_t, t\geq 0)$ the unique continuous increasing process such that :

$$R_t^{2\alpha} - L_t := N_t\,, t\geq 0\,, \quad \text{is a martingale.} \tag{1.4}$$

In the Brownian case, we have $d = 1$, $\alpha = 1/2$. Then we recover (1.2).
Thus, the corresponding Itô characteristic measure $\underline{n}_\alpha$ is, in fact, determined, and we shall give some descriptions of it below.
We shall denote by $(L_t^{(c)}, t \geq 0)$ the choice :

$$L_t^{(c)} = cL_t \,, \qquad \text{for some } c > 0. \tag{1.5}$$

Thus, the corresponding Itô measure $\underline{n}_\alpha^{(c)}$ satisfies :

$$\underline{n}_\alpha^{(c)} = \left(\frac{1}{c}\right) \underline{n}_\alpha \,, \tag{1.6}$$

since, by (1.1), which also holds of course for all the Bessel diffusions, with $0 < d < 2$, the product :

$$dL_t^{(c)} \, \underline{n}_\alpha^{(c)} \, (d\varepsilon) \quad \text{does not depend on } c.$$

One of our tasks in this Note is to present, in Section 5, the various constants c found in the literature, which, we hope, may be helpful in order to avoid trivial, but annoying, mistakes... But, first, in Section 2, we give an elementary stochastic calculus approach to formula (1.4); then, in Section 3, we examine some consequences of our choice for L.

1.3 Although the present work can hardly be considered as a research paper, but merely as a user's friendly note, we think its publication may be justified as the computations of the constants related to various choices of local times for BES $(-\alpha)$ often play a crucial role, if one deals simultaneously with several BES processes, e.g : relating one to another one by time changes, as in Biane-Yor [2], or trying to pass to the limit as, say, $\alpha \downarrow 0$, as in Donati-Martin-Yor [5].

2 A stochastic calculus approach

We now derive formula (1.4), and a number of properties of $(L_t, \; t \geq 0)$ by relying upon the definition of squares of Bessel processes, via the stochastic differential equation :

$$R_t^2 = 2 \int_0^t R_s d\beta_s + \delta t, \quad t \geq 0, \tag{2.1}$$

where $\delta \geq 0$ is given, and $(R_s, \; s \geq 0)$ is assumed to be an $\mathbb{R}_+$-valued process. It is by now well known that the equation (2.1), where $(\beta_s, \; s \geq 0)$ is a driving Brownian motion, admits a unique solution, which is strong (see, e.g. [15], Chap. IX).
Using stochastic calculus, we show the following

Theorem 2.1 *For $\delta = 2(1-\alpha), \; 0 < \alpha < 1$, the process $(R_t^{2\alpha}, \; t \geq 0)$ is a submartingale, whose Doob-Meyer decomposition is*

$$R_t^{2\alpha} = N_t + L_t, \quad t \geq 0, \tag{2.2}$$

with $(L_t, \; t \geq 0)$ a continuous increasing process, carried by the zeros of $(R_t, \; t \geq 0)$, and $(N_t, \; t \geq 0)$ a martingale which may be written :

$$N_t = 2\alpha \int_0^t R_s^{2\alpha - 1} d\beta_s, \quad t \geq 0. \tag{2.3}$$

Proof. Let $\varepsilon > 0$. We apply Itô's formula to the semimartingale : $\left(\left(\varepsilon + R_t^2\right)^\alpha, t \geq 0\right)$; thus we obtain :

$$\left(\varepsilon + R_t^2\right)^\alpha = \varepsilon^\alpha + 2\alpha \int_0^t \left(\varepsilon + R_s^2\right)^{\alpha - 1} R_s d\beta_s + 2\alpha(1-\alpha)\varepsilon \int_0^t \frac{ds}{\left(\varepsilon + R_s^2\right)^{2-\alpha}}, \quad t \geq 0. \quad (2.4)$$

It is not difficult to show, using dominated convergence that both the martingale part, and the increasing process part in (2.4) converge, in $L^2(\Omega)$, as $\varepsilon \to 0$. Moreover, concerning the Riemann integral, it is clear that, for any $\eta > 0$:

$$\varepsilon \int_0^t 1_{\{R_s \geq \eta\}} \frac{ds}{\left(\varepsilon + R_s^2\right)^{2-\alpha}} \to 0, \quad \text{as} \quad \varepsilon \to 0.$$

Thus, from (2.4), we deduce that, by letting $\varepsilon \to 0$:

$$R_t^{2\alpha} = 2\alpha \int_0^t R_s^{2\alpha - 1} d\beta_s + L_t, \quad t \geq 0,$$

where $(L_t, t \geq 0)$ is a continuous increasing process such that : $1_{\{R_t \neq 0\}} dL_t = 0$. ■

Corollary 2.2 *Let (R_t) as in Theorem 2.1. There exists a reflecting Brownian motion $(\rho_s, s \geq 0)$ such that :*

$$R_t^{2\alpha} = \rho_{A_t}, \quad \text{where } A_t = < N >_t = 4\alpha^2 \int_0^t R_s^{2(2\alpha - 1)} ds. \quad (2.5)$$

Proof. Define $\theta_u = \inf\{t \geq 0; A_t \geq u\}$. Then, from (2.2), we deduce :

$$\rho_u \stackrel{\text{def}}{=} R_{\theta_u}^{2\alpha} = -\gamma_u + L_{\theta_u}, \quad u \geq 0, \quad (2.6)$$

where, from (2.3) and the Dubins-Schwarz theorem, $(\gamma_u, u \geq 0)$ is a one dimensional Brownian motion. Since the increasing process $\left(L_{\theta_u}, u \geq 0\right)$ is supported on the set of the zeros of $(\rho_u, u \geq 0)$, the identity (2.6) may be considered as a particular case of Skorokhod's reflection equation. Hence, $L_{\theta_u} = \sup_{0 \leq v \leq u} \gamma_v$ and therefore $(\rho_u, u \geq 0)$ is a reflecting Brownian motion. ■

3 Some useful formulae related to the choice (1.4) of L

Proposition 3.1 *There exists a jointly continuous family $(L_t^x; x \geq 0, t \geq 0)$ of local times such that :*

1. $L_t^0 = L_t$;

2. *the occupation formula :*

$$\int_0^t h(R_s) ds = \frac{1}{\alpha} \int_0^\infty h(x) L_t^x \, x^{1-2\alpha} dx \quad (3.1)$$

holds for every Borel function $h : \mathbb{R}_+ \to \mathbb{R}_+$.

Proof. From the general theory of diffusion local times, we know that (3.1) holds with a certain constant $\mathcal{C}$, on the RHS, instead of $\frac{1}{\alpha}$ and our task is to show that $\mathcal{C} = \frac{1}{\alpha}$.
But, taking expectations on both sides of (3.1) (with $\mathcal{C}$ on the RHS, instead of $1/\alpha$), we get :

$$\int_0^t ds\, r_s(x) = \mathcal{C} x^{1-2\alpha} E[L_t^x]$$

where $r_s(x)$ is the density of R_s.
Recall that (see for instance [15] Chap. XI, p446) :

$$r_s(x) = \frac{2^\alpha s^{\alpha-1}}{\Gamma(1-\alpha)} x^{1-2\alpha} \exp\Big\{ -\frac{x^2}{2s}\Big\}, \quad s > 0, x \in \mathbb{R}. \tag{3.2}$$

Hence, from our choice (1.4), we get :

$$\mathcal{C} x^{1-2\alpha} E[R_t^{2\alpha}] \underset{x\to 0}{\sim} \int_0^t ds\, r_s(x). \tag{3.3}$$

But, since $R_u^2 \overset{(d)}{=} 2u\gamma_{d/2}$, where γ_k denotes a standard Gamma variable with parameter k, it easily follows from (3.3) that : $\mathcal{C} = \frac{1}{\alpha}$. ∎

Denote by $\tau_\ell \equiv \inf\{t : L_t > \ell\}, \ell \geq 0$ the right continuous inverse of $(L_t, t \geq 0)$.

Proposition 3.2 *The process $(\tau_\ell, \ell \geq 0)$ is a stable (α) subordinator, such that :*

$$E\left[\exp(-\lambda \tau_\ell)\right] = \exp\left(-\ell \frac{\Gamma(1-\alpha)}{\Gamma(1+\alpha)} 2^{-\alpha}\lambda^\alpha \right). \tag{3.4}$$

Moreover its Lévy measure $\nu(dt)$ is equal to $\dfrac{1}{2^\alpha \Gamma(\alpha)} \dfrac{dt}{t^{\alpha+1}} 1_{\{t>0\}}$, i.e.

$$E\left[\exp(-\lambda \tau_\ell)\right] = \exp\Big\{ -\ell \int_0^\infty \left(1 - e^{-\lambda t}\right) \frac{1}{2^\alpha \Gamma(\alpha)} \frac{dt}{t^{\alpha+1}} \Big\}. \tag{3.5}$$

Proof. 1) As in the previous proof, we know a priori that there exists a constant $\mathcal{C}'$ such that :

$$E\left[\exp(-\lambda \tau_\ell)\right] = \exp\left(-\ell\, \mathcal{C}' \lambda^\alpha\right) \tag{3.6}$$

and we wish to prove :

$$\mathcal{C}' = \frac{\Gamma(1-\alpha)}{2^\alpha \Gamma(1+\alpha)}. \tag{3.7}$$

However, by integration of both sides of (3.6) with respect to $(d\ell)$, we obtain :

$$E\left[\int_0^\infty dL_t \exp(-\lambda t)\right] = \frac{1}{\mathcal{C}' \lambda^\alpha}. \tag{3.8}$$

Again, from (1.4) the LHS is equal to :

$$\begin{aligned} E\left[\int_0^\infty dR_t^{2\alpha} \exp(-\lambda t)\right] &= E\left[R_1^{2\alpha}\right] \int_0^\infty \alpha\, t^{\alpha-1} dt \exp(-\lambda t) \\ &= 2^\alpha \frac{1}{\Gamma(1-\alpha)} \alpha\Gamma(\alpha) \frac{1}{\lambda^\alpha}, \end{aligned}$$

which yields (3.4).

2) An easy scaling argument shows that the Lévy measure ν of (τ_ℓ) is of the form : $\nu(dt) = C_\nu \frac{dt}{t^{\alpha+1}} 1_{\{t>0\}}$, where C_ν is a positive constant.

Integrating by parts, we have :

$$\int_0^\infty \left(1 - e^{-\lambda t}\right) \frac{dt}{t^{\alpha+1}} = \frac{\lambda}{\alpha} \int_0^\infty e^{-\lambda t} \frac{dt}{t^\alpha} = \frac{\lambda^\alpha}{\alpha} \Gamma(1-\alpha).$$

Consequently

$$C_\nu \frac{\lambda^\alpha}{\alpha} \Gamma(1-\alpha) = \frac{\Gamma(1-\alpha)}{\Gamma(1+\alpha)} 2^{-\alpha} \lambda^\alpha.$$

This implies that $C_\nu = \dfrac{1}{2^\alpha \Gamma(\alpha)}$. ∎

Furthermore, by scaling, there is the relation :

$$\text{for fixed } \ell\ , \quad \tau_\ell \overset{(d)}{=} (\ell/L_1)^{1/\alpha}. \tag{3.9}$$

If we denote by $(\theta_\ell(t), t \geq 0)$ the density of τ_ℓ, then :

$$\theta_\ell(t) = \frac{1}{\pi t} \sum_{k=1}^\infty \frac{(-1)^{k+1} \Gamma(\alpha k + 1)}{k!} \left(\frac{\gamma}{t^\alpha}\right)^k \sin(k\pi\alpha) \quad (t \geq 0)$$

with :

$$\gamma := \ell\, \frac{\Gamma(1-\alpha)}{\Gamma(1+\alpha)}\, 2^{-\alpha} \qquad \text{(see [9], p. 142).}$$

In particular :

$$\theta_\ell(t) \underset{t\to\infty}{\sim} \gamma\, \frac{\Gamma(\alpha+1)}{\pi t^{\alpha+1}} \sin \pi\alpha = \frac{\ell}{\Gamma(\alpha)}\, \frac{2^{-\alpha}}{t^{\alpha+1}}\ , \tag{3.10}$$

because $\Gamma(\alpha)\Gamma(1-\alpha) = \dfrac{\pi}{\sin \pi\alpha}$, (cf [8], p. 3).

We now discuss similar properties for the density $(p_t(\ell), \ell \geq 0)$ of L_t. The scaling property yields :

$$L_t \overset{(d)}{=} t^\alpha L_1\,. \tag{3.11}$$

Proposition 3.3 *The following relation between $\theta_\ell(t)$ and $p_t(\ell)$ holds :*

$$\theta_\ell(t) = \frac{\ell\alpha}{t^{\alpha+1}} \, p_1\left(\frac{\ell}{t^\alpha}\right).$$

Moreover :

$$p_1(0) = \frac{2^{-\alpha}}{\Gamma(1+\alpha)}. \tag{3.12}$$

Proof. The relation between $\theta_\ell(t)$ and $p_t(\ell)$ follows directly from (3.9).
Taking the limit of

$$p_1\left(\frac{\ell}{t^\alpha}\right) = \frac{t^{\alpha+1}}{\ell\alpha}\theta_\ell(t),$$

as $t \to \infty$, and using (3.10), we obtain :

$$\lim_{t\to\infty} p_1\left(\frac{\ell}{t^\alpha}\right) = \lim_{t\to\infty} \frac{t^{\alpha+1}}{\ell\alpha} \frac{\ell}{\Gamma(\alpha)} \frac{2^{-\alpha}}{t^{\alpha+1}} = \frac{2^{-\alpha}}{\Gamma(1+\alpha)}.$$

This shows (3.12). ■

The positive moments of $L_1 \overset{(d)}{=} (\tau_1)^{-\alpha}$ have a nice expression.

Proposition 3.4 *Let m be a positive real. Then*

$$E\left[(L_1)^m\right] = \frac{\Gamma(1+m)}{\Gamma(1+\alpha m)} \left(\frac{2^\alpha \Gamma(1+\alpha)}{\Gamma(1-\alpha)}\right)^m. \tag{3.13}$$

Proof. 1) It easily follows from the usual computation of negative moments of a $\mathbb{R}_+$-valued r.v. that :

$$E\left[\frac{1}{(\tau_1)^{\alpha m}}\right] = \frac{1}{\Gamma(\alpha m)} \int_0^\infty du\, u^{\alpha m - 1} E\left[e^{-u\tau_1}\right]$$

and it now remains to use (3.4).
2) An alternative proof of (3.13) consists in using the fact that, for S a standard exp(1) distributed variable, independent of $(L_t, t \geq 0)$, one has $L_S \overset{(d)}{=} S^\alpha L_1$, and L_S is distributed as an exp($\mathcal{C}'$) variable, with $\mathcal{C}'$ given by (3.7).
3) For m an integer, the referee suggested another proof based on Kac's recursive moment formula :

$$E\left[(L_t)^m\right] = mE\left[\int_0^t E\left[(L_{t-s})^{m-1}\right] dL_s\right]. \tag{3.14}$$

Assuming (3.14) for a moment, one recovers (3.13) by recurrence, from the scaling property of $(L_t, t \geq 0)$.

■

We now give a proof of formula (3.14).
This identity may be proved using the two first formulae of [7], p533, with $\varphi(x) = 1$.
It seems interesting to give a direct proof of (3.14) in a more general setting than Bessel processes. Suppose that $(L_t, t \geq 0)$ is the local time process at 0 of a diffusion started at 0, such that $L_\infty = \infty$ a.s. We claim that :

$$E[G(L_t)] = E\big[\int_0^t E[g(L_{t-s})]dL_s\big], \tag{3.15}$$

where $g : \mathbb{R}_+ \to \mathbb{R}_+$ is Borel and $G(x) = \displaystyle\int_0^x g(t)dt,\ x \geq 0$.
Obviously, (3.15) is an extension of (3.14) since taking $g(x) = mx^{m-1}$ in (3.15) we recover (3.14).
To prove (3.15), we introduce the two following positive measures on $\mathbb{R}_+$:

$$\mu_1(t; A) = E\big[\int_0^t 1_A(L_s)dL_s\big], \quad \mu_2(t; A) = E\big[\int_0^t P(L_{t-s} \in A)dL_s\big], \quad t \geq 0,$$

where A is a generic Borel set included in $\mathbb{R}_+$.
It is clear that $\mu_1(t; \cdot) = \mu_2(t; \cdot)$ if and only if (3.15) holds. Therefore (3.15) is equivalent to : $\theta_1(\lambda, a) = \theta_2(\lambda, a)$, for any $\lambda, a > 0$, where

$$\theta_i(\lambda, a) = \int_0^\infty e^{-\lambda t}\mu_i(t; e^{-a\cdot})dt.$$

Let (τ_u) be the right-inverse of (L_t) (i.e. $\tau_u = \inf\{s \geq 0; L_s > u\}$) and ψ be the Lévy exponent of (τ_u) :

$$E\big[e^{-\lambda\tau_u}\big] = e^{-u\psi(\lambda)}, \quad u, \lambda > 0. \tag{3.16}$$

We first compute $\theta_1(\lambda, a)$:

$$\theta_1(\lambda, a) = \int_0^\infty dte^{-\lambda t}E\big[\int_0^t e^{-aL_s}dL_s\big] = \frac{1}{\lambda}E\big[\int_0^\infty e^{-aL_s-\lambda s}dL_s\big].$$

Setting $u = L_s$ and using (3.16), we obtain :

$$\theta_1(\lambda, a) = \frac{1}{\lambda}E\big[\int_0^\infty e^{-au-\lambda\tau_u}du\big] = \frac{1}{\lambda}\int_0^\infty e^{-u(a+\psi(\lambda))}du = \frac{1}{\lambda(\psi(\lambda) + a)}.$$

Similarly, we express $\theta_2(\lambda, a)$ in terms of $\psi(\lambda)$:

$$\theta_2(\lambda, a) = \int_0^\infty dte^{-\lambda t}E\big[\int_0^t E[e^{-aL_{t-s}}]dL_s\big].$$

Using Fubini's theorem we get :

$$\theta_2(\lambda, a) = E\big[\int_0^\infty e^{-\lambda s}dL_s\big]\ E\big[\int_0^\infty e^{-aL_u-\lambda u}du\big].$$

Since $E\big[\displaystyle\int_0^\infty e^{-\lambda s}dL_s\big] = E\big[\int_0^\infty e^{-\lambda\tau_u}du\big]$, then (3.16) implies that :

$$\theta_2(\lambda, a) = \frac{1}{\psi(\lambda)}E\big[\int_0^\infty e^{-aL_u-\lambda u}du\big].$$

Using excursions theory, we obtain :

$$\begin{aligned}\theta_2(\lambda,a) &= \frac{1}{\psi(\lambda)}E\Big[\sum_{l>0}\int_{\tau_{l-}}^{\tau_l} e^{-aL_u-\lambda u}du\Big] = \frac{1}{\psi(\lambda)}E\Big[\sum_{l>0} e^{-al-\lambda\tau_{l-}}\frac{1-e^{-\lambda(\tau_l-\tau_{l-})}}{\lambda}\Big] \\ &= \frac{1}{\lambda\psi(\lambda)}\Big(\int_0^\infty e^{-al}E[e^{-\lambda\tau_l}]dl\Big)\psi(\lambda).\end{aligned}$$

Using (3.16) and (3.5), we have :

$$\theta_2(\lambda,a) = \frac{1}{\lambda\psi(\lambda)}\Big(\int_0^\infty e^{-l(a+\psi(\lambda))}dl\Big)\psi(\lambda) = \frac{1}{\lambda(a+\psi(\lambda))}.$$

This proves that $\theta_1(\lambda,a)=\theta_2(\lambda,a)$. Hence, formula (3.15) has been proved.
Likewise, the moments of l_1, the local time of the standard Bessel bridge, with dimension $2(1-\alpha)$ may be computed explicitly as the following proposition shows.

Proposition 3.5 *For m, any positive real, we have :*

$$E\left[(l_1)^m\right] = \frac{\Gamma(1+m)\Gamma(\alpha)}{\Gamma\big(\alpha(1+m)\big)}\left(\frac{2^\alpha\Gamma(1+\alpha)}{\Gamma(1-\alpha)}\right)^m. \tag{3.17}$$

The main difference with (3.13) is that in the denominator of (3.17), we find $\Gamma\big(\alpha(1+m)\big)$, whereas in that of (3.13), we have $\Gamma\big((1+\alpha)m\big)$.

Proof of Proposition 3.5
Let $g=\sup\{t<1, R_t=0\}$.
We have :

$$L_1 = L_g = g^\alpha l_1,$$

by scaling, using the fact that $\big(\frac{1}{\sqrt{g}}R_{ug}, 0\le u\le 1\big)$ is a standard Bessel bridge independent of g.
Next, we use the fact that $g \overset{(d)}{=} \beta(\alpha,1-\alpha)$. Consequently :

$$E\left[(L_1)^m\right] = E\big[g^{\alpha m}\big]E\left[(l_1)^m\right] = \frac{\Gamma\big(\alpha(1+m)\big)}{\Gamma(1+\alpha m)\Gamma(\alpha)}E\left[(l_1)^m\right].$$

Comparing this formula with (3.13), we obtain (3.17).

■

4 Descriptions of $\underline{n}_\alpha$

Traditionally, following D. Williams' descriptions of Itô's measure for Brownian motion, (see e.g : Williams [17] and Rogers [16]), there are two descriptions for $\underline{n}_\alpha$, which are obtained by disintegrating $\underline{n}_\alpha$ with respect to $M=\sup_{u\le v(\varepsilon)}\varepsilon(u)$ or with respect to $V(\varepsilon)=\inf\{t>0;\varepsilon(t)=0\}$, where ε denotes the generic excursion and $V(\varepsilon)$ its lifetime.
These descriptions are (see [2], and [11]) :

4.1 Disintegration with respect to M

(i) $\underline{n}_\alpha(M \geq a) = a^{-2\alpha}$; (4.1)

(ii) Conditionally on $M = a, \{\varepsilon(u), u \leq V(\varepsilon)\}$ may be obtained by putting two independent BES (α) processes back to back, up to their first hitting times of a,

where BES(α) denotes the Bessel process started at level 0, with positive index α, or dimension $d' = 2(1+\alpha) > 2$.

4.2 Disintegration with respect to V

(i) $\underline{n}_\alpha(V \in dv) = \dfrac{dv}{2^\alpha \Gamma(\alpha) v^{\alpha+1}}$; (4.2)

(ii) Conditionally on $V = v,$ the process $(\varepsilon(u), u \leq v)$ is distributed as a standard BES (α) bridge, with length v.

In fact, a simple look at either (4.1) (i), or (4.2)(i), as well as the corresponding formulae in Biane-Yor [2], shows that : $\underline{n}_\alpha = \hat{n}_\alpha$, where $\hat{n}_\alpha$ is the notation for Itô measures chosen in [2]. Thus, it suffices to show (4.1) (i) : on one hand, from excursion theory, one has :

$$P\big\{\max_{u \leq \tau_\ell} R_u < a\big\} = \exp\left\{-\ell \underline{n}_\alpha(M \geq a)\right\}. \tag{4.3}$$

On the other hand, trivially,

$$P\left\{\max_{u \leq \tau_\ell} R_u < a\right\} = P\left\{\tau_\ell < T_a\right\} = P\left\{L_{T_a} > \ell\right\}. \tag{4.4}$$

However, L_{T_a} is exponentially distributed, with mean $E[L_{T_a}] = a^{2\alpha}$, from our choice of L. Comparing (4.3) and (4.4), we obtain (4.1)(i).

Remark 4.1 *We take this opportunity to correct an annoying mistake in (3.h) [2], where the constant* $\frac{1}{4\mu(1-\mu)}$ *should be changed in* $\frac{1}{2\mu(1-\mu)}$ *(see also a similar remark before paragraph (3.3) in [5]).*

5 The different choices of c in the literature

As we scanned the literature about excursion theory and/or choice of local times for BES $(-\alpha)$, we have come across the following papers :

1. Pitman-Yor ([12], [14], [13]) : in these three papers, the results do not depend on the local time normalization;
2. Gradinaru-Roynette-Vallois-Yor ([7]; p.538);
3. Borodin-Salminen ([3], p.80);

4. Chaumont-Yor : Exercises in Probability ([4]), Exercise 4.19, p.114;
5. Barlow-Pitman-Yor ([1], p.299);
6. Biane-Yor ([2], p.44, (3.f));
7. Pitman-Yor ([11], p.300);
8. Donati-Martin-Yor ([5])
9. Molchanov-Ostrovskii ([10])

In most of these papers from 2. to 9., the authors use a particular density of occupation formula :

$$\int_0^t h(R_s)ds = \gamma \int_0^\infty h(x) L_t^{x(c)} x^\beta dx \,,$$

which, when compared with (3.1) yields :

$$L_t^{x(c)} = \frac{1}{\alpha\gamma} x^{(1-2\alpha-\beta)} L_t^x \tag{5.1}$$

(in many cases, $\beta = 1 - 2\alpha$, so that : $L_t^{x(c)} = \frac{1}{\alpha\gamma} L_t^x$).

We now present a table for the formulae (5.1) obtained from the papers 2. to 9.

[7]	$L_t^{x(c)} = \dfrac{1}{2\alpha} L_t^x$	[3]	$L_t^{x(c)} = \dfrac{1}{\alpha} x^{1-2\alpha} L_t^x$
[4]	$L_t^{(c)} = \dfrac{\Gamma(1-\alpha)2^{-\alpha}}{\Gamma(1+\alpha)} L_t$	[1]	$L_t^{x(c)} = \dfrac{\Gamma(1-\alpha)}{\Gamma(1+\alpha)} 2^{-\alpha} L_t^x$
[2] (p.44)	$L_t^{x(c)} = \dfrac{1}{2\alpha(1-\alpha)} L_t^x$	[11] (p.300)	$L_t^{x(c)} = \dfrac{1}{\alpha} x^{1-2\alpha} L_t^x$
[5]	$L_t^{x(c)} = \dfrac{1}{\alpha(2^\alpha\Gamma(\alpha))^2} L_t^x$	[10]	$L_t^{(c)} = 2^{-\alpha}\Gamma(1+\alpha) L_t$

Acknowledgement : We thank the referee for his constructive suggestions which helped to improve our paper.

References

[1] M. Barlow, J. Pitman, and M. Yor. Une extension multidimensionnelle de la loi de l'arc sinus. In *Séminaire de Probabilités, XXIII*, volume 1372 of *Lecture Notes in Math.*, pages 294–314. Springer, Berlin, 1989.

[2] Ph. Biane and M. Yor. Valeurs principales associées aux temps locaux browniens. *Bull. Sci. Math. (2)*, 111(1):23–101, 1987.

[3] A. N. Borodin and P. Salminen. *Handbook of Brownian motion—facts and formulae.* Probability and its Applications. Birkhäuser Verlag, Basel, second edition, 2002.

[4] L. Chaumont and M. Yor. *Exercises in probability*, volume 13 of *Cambridge Series in Statistical and Probabilistic Mathematics*. Cambridge University Press, Cambridge, 2003. A guided tour from measure theory to random processes, via conditioning.

[5] C. Donati-Martin and M. Yor. Some explicit Krein representations of certain subordinators, including the Gamma process. *To appear in Publ. RIMS, Kyoto*, 2006.

[6] W. Feller. *An introduction to probability theory and its applications. Vol. II.* Second edition. John Wiley & Sons Inc., New York, 1971.

[7] M. Gradinaru, B. Roynette, P. Vallois, and M. Yor. Abel transform and integrals of Bessel local times. *Ann. Inst. H. Poincaré Probab. Statist.*, 35(4):531–572, 1999.

[8] N. N. Lebedev. *Special functions and their applications.* Dover Publications Inc., New York, 1972. Revised edition, translated from the Russian and edited by Richard A. Silverman, Unabridged and corrected republication.

[9] E. Lukacs. *Characteristic functions.* Hafner Publishing Co., New York, 1970. Second edition, revised and enlarged.

[10] S. A. Molčanov and E. Ostrovskiĭ. Symmetric stable processes as traces of degenerate diffusion processes. *Teor. Verojatnost. i Primenen.*, 14:127–130, 1969.

[11] J. Pitman and M. Yor. Decomposition at the maximum for excursions and bridges of one-dimensional diffusions. In *Itô's stochastic calculus and probability theory*, pages 293–310. Springer, Tokyo, 1996.

[12] J. Pitman and M. Yor. Random Brownian scaling identities and splicing of Bessel processes. *Ann. Probab.*, 26(4):1683–1702, 1998.

[13] J. Pitman and M. Yor. Laplace transforms related to excursions of a one-dimensional diffusion. *Bernoulli*, 5(2):249–255, 1999.

[14] J. Pitman and M. Yor. On the distribution of ranked heights of excursions of a Brownian bridge. *Ann. Probab.*, 29(1):361–384, 2001.

[15] D. Revuz and M. Yor. *Continuous martingales and Brownian motion*, volume 293 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, third edition, 1999.

[16] L. C. G. Rogers. Williams' characterisation of the Brownian excursion law: proof and applications. In *Seminar on Probability, XV (Univ. Strasbourg, Strasbourg, 1979/1980) (French)*, volume 850 of *Lecture Notes in Math.*, pages 227–250. Springer, Berlin, 1981.

[17] D. Williams. *Diffusions, Markov processes, and martingales. Vol. 1.* John Wiley & Sons Ltd., Chichester, 1979. Foundations, Probability and Mathematical Statistics.